On the trace of $J^k_- J^l_z J^k_+$

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ADDENDUM

On the trace of $J_-^k J_z^l J_+^k$

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Abstract. It is shown that the trace of $J_-^k J_z^l J_+^k$ is given by

$$\frac{(2j)!k!}{(2j-k)!} j^l F(-2j+k, 1+k; -2j, \xi_0),$$

where ξ_0 is approximately of the form $\xi_0 = (1-1/j)^l$.

1. Introduction

Recently there has been considerable interest in deriving closed-form expressions for the traces of the products of angular momentum operators. Despite three new ways (De Meyer and Van den Berghe 1978, Rashid 1979, Ullah 1980) of looking at this problem, no closed-form expression is found for the trace of $J_-^k J_z^l J_+^k$. As shown in these references one could either derive recursive relations or express it as a double summation. The purpose of this addendum is to show that an approximate expression for this trace can be obtained in terms of the hypergeometric function, which is exact in the two limits $l = 0$ and $l \to \infty$.

2. Formulation

We start from the following closed-form expression (Ullah 1980), involving J_-^k, $\exp(\theta J_z)$, J_+^k,

$$\mathrm{Tr}[J_-^k \exp(\theta J_z) J_+^k] = \frac{(2j)!\,k!}{(2j-k)!} \exp(j\theta) F(-2j+k, 1+k; -2j; \exp(-\theta)), \tag{1}$$

where $F(a, b; c; z)$ denotes the hypergeometric function (Abramowitz and Stegun 1965). Denoting the trace of $J_-^k J_z^l J_+^k$ by $\mathrm{Tr}(k, l)$ and using contour integration, we obtain

$$\mathrm{Tr}(k, l) = \frac{(2j)!\,k!\,l!}{(2j-k)!} \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \frac{\exp(jz)}{z^{l+1}} F(-2j+k, 1+k; -2j; \exp(-z)). \tag{2}$$

Expression (2) can be rewritten as

$$\mathrm{Tr}(k, l) = \frac{(2j)!\,k!}{(2j-k)!} j^l \left(\frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \frac{\exp(jz)}{z^{l+1}} F(-2j+k, 1+k; -2j; \exp(-z)) \right) \times \left(\frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \frac{\exp(jz)}{z^{l+1}} \right)^{-1}. \tag{3}$$

Denoting $\exp(-z)$ by ξ and expanding the hypergeometric function around some point ξ_0, we obtain from expression (3)

$$\mathrm{Tr}(k,l)=\frac{(2j)!\,k!}{(2j-k)!}j^l F(-2j+k,1+k;-2j;\xi_0)$$
$$\times\left\{1+\frac{F'}{F}\left[\left(1-\frac{1}{j}\right)^l-\xi_0\right]+\frac{1}{2!}\frac{F''}{F}\left[\left(1-\frac{2}{j}\right)^l-2\left(1-\frac{1}{j}\right)^l\xi_0+\xi_0^2\right]+\dots\right\},\tag{4}$$

where F', F'', etc are derivatives of the hypergeometric function at ξ_0.

We first note from expression (4) that $\mathrm{Tr}(k,l)$ is given exactly by

$$\mathrm{Tr}(k,l)=\frac{(2j)!\,k!}{(2j-k)!}j^l F(-2j+k,1+k;-2j;\xi_0),\tag{5}$$

with $\xi_0=1$ when $l=0$, and $\xi_0\to 0$ when $l\to\infty$.

We shall now find a value of ξ_0 for other values of l, so that the trace is still given by expression (5) to a good approximation.

The simplest approximation which one can make is to make the coefficient of F' vanish. This is the same kind of approximation which one makes in the theory of probability (Papoulis 1965) to evaluate certain integrals approximately. ξ_0 is then given by

$$\xi_0=(1-1/j)^l.\tag{6}$$

From expressions (4) and (6) it is obvious that the absolute value of the next term in the expansion (multiplying F''/F, which represents the fluctuation) is quite small, and therefore expressions (5) and (6) provide a good approximation for $\mathrm{Tr}(k,l)$.

For illustrative purposes we calculate the trace of $(J_-J_z^4J_+)$ for the case of $j=\frac{5}{2}$. The approximate value of

$$\mathrm{Tr}(k,l)\Big/\frac{(2j)!\,k!}{(2j-k)!}j^l$$

using expressions (5) and (6) is 1·241, compared with its exact value of 1·342. The absolute value of the coefficient multiplying F''/F is 0·008, indicating that approximation (6) is a fairly good one.

3. Concluding remarks

The trace of the operator $J_-^kJ_z^lJ_+^k$ is written as a dominant part plus a small fluctuating part. The dominant part is given by expression (5), ξ_0 being determined from the vanishing of the coefficient of F'/F. The first remark which we would like to make is that, since F satisfies the second-order differential equation, we can also determine ξ_0 by expressing F'' in terms of F' and F. This gives a cubic equation to determine ξ_0, but does not much improve the accuracy of $\mathrm{Tr}(k,l)$. The second remark is that the contour integrals in expression (3) can be evaluated by the method of steepest descent (ter Haar 1954) when j, l are large, $l\gg j$. This does not give anything new, except that ξ_0 given by expression (6) can now be written as $\xi_0=\exp(-l/j)$.

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